Turbulent domain stabilization in annular flows

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(Received 17 May 1994)

We point out a mechanism for stabilizing expanding turbulent domains in annular flows. This nonlocal mechanism is explained within the context of a Ginzburg-Landau equation for a real amplitude. The expression for the nonlocal term can be derived by analogy with existing calculations in Taylor-Couette flow for Taylor vortices. Numerical results are compared with experiment.

PACS number(s): 47.27.Eq, 47.20.Ky

Turbulence can manifest itself in coherent structures. Spectacular examples of turbulent structures coexisting with laminar flow are provided by slugs in pipe flow, Emmons spots in boundary layer flow, and spiral turbulence in Taylor-Couette flow. It has been suggested that a model equation of the Ginzburg-Landau (GL) [1] type with subcritical behavior can provide a convenient starting point for discussing such situations. The reason is that in some range of parameters, this equation supports two linearly stable solutions, one of zero amplitude (the laminar phase) and one of finite amplitude (the turbulent phase).

The difficulty with the subcritical GL equation is that as long as its coefficients are real, any finite size domain of the turbulent state surrounded by the laminar one is unstable [3–5]. At one particular, critical value of the control parameter μ [see Eq. (1)] a state of large extent, with sharp boundaries, may be very long lived [5]. However, this is not the typical situation, for which the control parameter is determined by some externally imposed constraints, and one wants to understand how an expanding (or contracting) domain can stabilize for some finite range of parameters. This difficulty with the real GL equation is remedied when the coefficients become complex. In that case stable, pulselike solutions of complex amplitude exist [3], and their behavior has been discussed [3–5] and argued to be relevant to describing localized structures in binary convection.

We discuss here a different mechanism relevant in cases of annular flows. The starting point is the subcritical GL equation with real coefficients. Stabilization of domains, however, occurs not because of complex coefficients, but through a nonlocal term.

It has been argued before [2] that backflow can lead to the stabilization of turbulent spots and their coexistence with laminar domains. The remarks referred, in particular, to the case of spiral turbulence in the Taylor-Couette flow [6]. Here, in a section of the cylinders perpendicular to their common axis the laminar and turbulent domain coexit, and in the appropriate range of control parameters the situation is hysterectic [7]: within the laminar domain a turbulent perturbation grows to a spot which fills close to half the available space in a cylinder section perpendicular to the axis.

Previous work on amplitude equations, which deals not with spiral turbulence, but with the amplitudes of Taylor vortices close to the threshold of their occurrence [8,10,9] indicates how backflow may arise. In particular, Hall [10] has shown that (in the small gap limit) one is led to the introduction of an azimuthal pressure gradient, which adds to the azimuthal velocity a Poiseuille component, leading in turn to a nonlocal term in the amplitude equation. Before giving details about how a similar calculation should be relevant to the coexistence of laminar and turbulent domains, let us show numerically and also explain how a certain nonlocal term in a GL equation with real coefficients can lead to a stable domain structure, whereas in the absence of such a term none exists, as mentioned above.

The equation without the nonlocal term is the usual subcritical GL equation for a complex amplitude A with real coefficients (written in one dimension and after convenient rescaling):

$$\partial A/\partial t=\mu A+A_{xx}+\beta|A|^2A+\gamma|A|^4A\ . \tag{1}$$

The phase of A does not play any role and in the subsequent discussion A may be as well considered a real function. Here μ is the control parameter, the coefficients β and γ are, respectively, positive and negative. The behavior of $|A|$ as a function of μ is shown in Fig. 1. For μ negative between 0 and $\beta^2/4\gamma$ the two stationary and uniform states $A=0$ and A^* are linearly stable. However, they may be unstable to finite perturbations. What happens is determined by the right hand side of Eq. (1) which derives from a potential V such that

$$\partial A/\partial t=A_{xx}-\partial V/\partial A^c\ , \tag{2}$$

where A^c is the complex conjugate of A. The potential has two minima in the range of control parameter mentioned above. When these are equal, at $\mu=\mu_{cr}=3\beta^2/16\gamma$, the two corresponding states, $A=0$ and A^* coexist. When they are not equal, one state invades the other one, the $A=0$ state the A^* one for $\mu<\mu_{cr}$, and vice versa for $\mu>\mu_{cr}$ [2,4].

For our purposes we interpret A as the amplitude of turbulent fluctuations, and thus $A=0$ represents the laminar state and $A=A^*$ the turbulent state. Now the modification to Eq. (1) which leads to turbulent domain stabilization, consists in introducing the nonlocal term

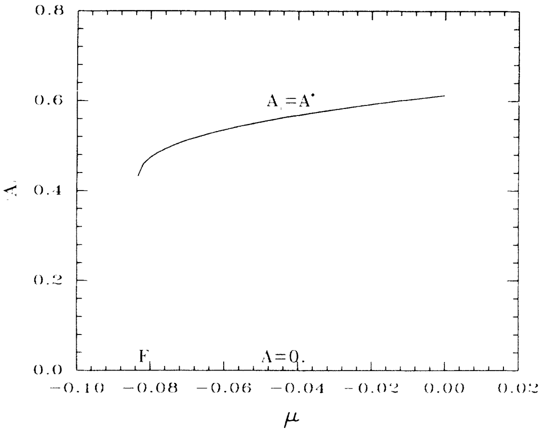

FIG. 1. Diagram indicating the variation of $|A|$ with control parameter μ. The $A=0$ and A^* states coexist from $\mu=0$ to F. The curve is drawn for $\beta=0.5$ and $\gamma=-0.75$, for which $F=-\frac{1}{12}=-0.0833$. $A=A^*$ corresponds to the full line. The line with crosses corresponds to a linearly unstable state.

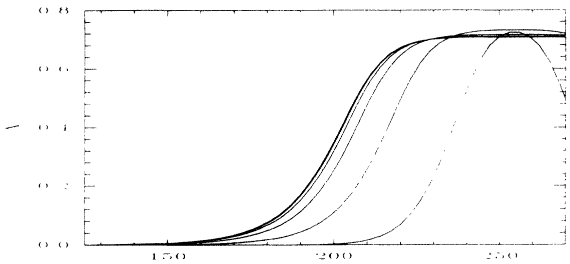

FIG. 2. Expansion of a turbulent domain in time (only a fraction of the full domain is shown). The abscissa indicates system size in lattice points. Distance between consecutive lattice points is $\delta x=\frac{1}{3}$. The ordinate shows $|A|$. There are six curves. Successive curves are equidistant in time, by 3600 time steps where each time step is $\delta t=0.02$, the first curve being obtained this way from the starting configuration, for which the domain is of width equal to 30 lattice points. Values of the parameters are $\beta=0.5$, $\gamma=-0.75$, and $\mu=\mu_{\rm cr}+0.04=-1/16+0.04$.

$$\mathcal{J}=(1/L)\int_0^L|A|^2dx\ , \tag{3}$$

such that the modified equation reads

$$\partial A/\partial t=\mu A+A_{xx}+\beta(|A|^2-\mathcal{J})A+\gamma|A|^4A\ . \tag{4}$$

Here, L denotes the perimeter of the annular flow region. (The calculation is done for a circular region in the small gap limit.) Since the contribution to $\mathcal{J}$ comes only from the turbulent domain, the value of $\mathcal{J}$ is smaller than 1 (A^* has values close to 1). Neglecting domain boundaries, one thus has $\mathcal{J}\approx A^{*2}l/L$, where l is the extent of the turbulent domain (in one space dimension).

There is no reason why the (dimensionless) coefficient in front of $\mathcal{J}$ in (4) should be taken exactly equal to 1. Our choice is one of simplicity, and is also in agreement with the result of the model calculation performed further on [cf. Eq. (12)].

We show in Fig. 2 what is happening. The starting point is a domain size of $l=30$ (with $L=600$) of A^*, embedded in a laminar domain $A=0$, for values of μ, β, γ such that $\mu>\mu_{\rm cr}$. For $\mathcal{J}=0$ the turbulent domain would expand into the laminar one and completely fill it. However, here, as soon as the domain expands, the contribution to $\mathcal{J}$ increases, the effective β, whose value is very roughly equal to $\beta(1-l/L)$ (see end of preceding paragraph), diminishes, and, therefore, the effective $\mu_{\rm cr}$ decreases in absolute value until its value is equal to that of μ, at which time the expansion stops. The opposite phenomenon happens when the initial domain is correspondingly larger, in which case initially μ is smaller than the effective $\mu_{\rm cr}$. (Both quantities are negative.) The turbulent domain then shrinks to the same final size (for the same μ, β, γ of course). Take that final domain, and assume one makes it larger: $\mathcal{J}$ then increases, the effective β decreases in absolute value, the corresponding effective $\mu_{\rm cr}$ moves to the right of μ (see Fig. 1), and thus the laminar domain expands, or equivalently the turbulent one shrinks back to its initial value. The same regulatory mechanism acts if one starts by shortening the domain. Thus the turbulent domain at the point where the effective $\mu_{\rm cr}$ is equal to μ is stable. Since the effective $\mu_{\rm cr}$ depends on the ratio l/L only, not on l or L separately, it is the relative size of the turbulent domain which, for a given set of parameters, is fixed.

Let us now discuss the velocity of an expanding domain. In Fig. 3 is shown how the length of the domain, relative to its initial size, increases as a function of time. The initial growth is practically linear. As, however, the effective $\mu_{\rm cr}$ approaches the given value of μ, the expansion slows and the size becomes constant. From the initial linear rise one can define a rate of expansion. This rate of expansion depends on how close the initial size is as compared to the final one. Figure 4 shows that this initial rate of expansion itself varies nearly linearly as a function of distance from threshold. This linear variation is in qualitative agreement with theoretical results on Eq. (1) (for which however there is no saturation.) It is noteworthy that the results of Figs. 3 and 4 are qualitatively similar to those which Dauchot and Daviaud [11] have obtained for the behavior of stable turbulent spots in plane Couette flow, and thus appear as generic properties of a subcritical GL amplitude equation. The curve in Fig. 4 does not pass through the origin, because, for the

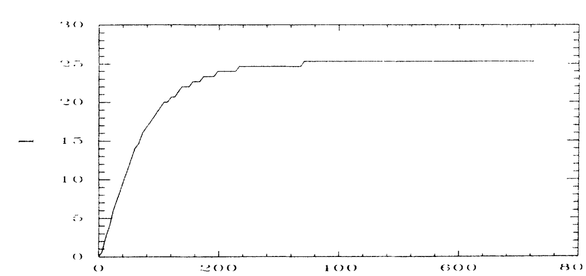

FIG. 3. Size l of an expanding turbulent domain as a function of time. Initial size has been subtracted. Time and length are in real units. Parameter values are the same as for Fig. 2.

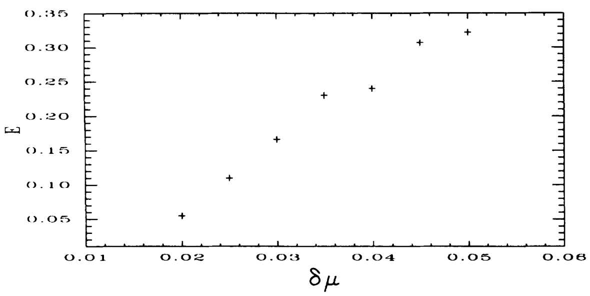

FIG. 4. Expansion rate E of a domain as a function of distance to threshold. The abscissa represents $\delta\mu=\mu-\mu_{cr}$. Parameter values are the same as for Fig. 1. The size of the original domain is 30 in lattice units.

given initial size (30 in lattice units), the domain actually starts shrinking when the abscissa goes below 0.02.

Now it remains to be shown how a nonlocal term arises in annular flow, in the first place, which then leads to a transformation of Eq. (1) into the form given by Eq. (2). We are following Hall's [10] lead. It is easily shown [8–10] that in the small gap limit (width of annulus much smaller than average radius), the equation of mass conservation takes the "Cartesian" form:

$$\partial u/\partial y+\partial v/\partial x=0\ , \tag{5}$$

where y and x are, respectively, the appropriately scaled radial and azimuthal variables, and u and v the dimensionless radial and azimuthal velocity components. y varies in the interval [0,1] and x, which is periodic, in the interval $[0,L]$.

We assume that v takes the form

$$v=|A|^2(x)f(y)\ . \tag{6}$$

Such a form represents the average flow velocity in both turbulent and laminar regions, and is proportional to the square of the amplitude of turbulent fluctuations. It seems plausible that the only large scale change introduced by turbulence into a laminar, circularly symmetric, flow is the dependence of A on the azimuthal variable. In Hall's work [10] the above form of v arises in an amplitude expansion close to the threshold for the occurrence of Taylor vortices. Here, the above form is taken as a plausible starting point.

Now, as soon as v acquires a x dependence, a radial velocity component u exists, related to v by Eq. (5). Our description is incomplete, however, because u thus defined cannot satisfy the no slip boundary condition at $y=0$ and 1, which requires that $\int_0^1(\partial u/\partial y)dy=0$, as long as there is a nonzero mass flux through an annular cross section. The remedy is that there exists an azimuthal pressure gradient such that the equation satisfied by v is the following:

$$\partial^2 v/\partial^2 y=-dp/dx+|A|^2 d^2 f/d^2 y\ . \tag{7}$$

In the small gap limit the second derivative relative to y is the dominant term in the laplacian of v. Moreover, one indeed expects that turbulent fluctuations will engender an azimuthal pressure gradient.

Integrating Eq. (7) one arrives at

$$v=-(\tfrac{1}{2})(y^2-y)dp/dx+|A|^2(x)f(y)\ . \tag{8}$$

Using this expression for v, and Eq. (5), the boundary condition on u (given above) leads to

$$d^2p/d^2x=-12Hd^2|A|^2/d^2x\ , \tag{9}$$

where

$$H=\int_0^1 f(y)dy\ . \tag{10}$$

Equation (9) for pressure can now be integrated. Imposing the condition that $p(0)=p(L)$ one finds

$$dp/dx=12H(\mathcal{J}-|A|^2)\ , \tag{11}$$

where $\mathcal{J}$ is the integral given by expression (3). The integral term represents the effect on pressure of turbulent fluctuations, via an expression of the Reynolds stress type.

Now, for v, using (8), one has

$$v=|A|^2(x)f(y)+6H(|A|^2-\mathcal{J})(y^2-y)\ . \tag{12}$$

The coefficient of the Poiseuille y dependent term is exactly the term which appears in the amplitude equation, namely, Eq. (4).

In the case of a small parameter expansion [10], the preceding equation then leads, through a solvability condition, to an amplitude equation such as Eq. (4) with the correction due to the presence of a nonlocal term. In our case, our discussion above can only make it plausible that, whenever a subcritical GL amplitude equation is used for annular flows, to describe the coexistence of turbulent and laminar domains, a nonlocal term of the type discussed occurs. The flow has to be annular, because the derivation of the expression for the nonlocal term depends crucially on the periodicity of pressure. For such cases, however, the mechanism for turbulent spot stabilization proposed here appears a natural consequence of the hydrodynamics which describe the flow.

This work was supported by the U. S. Department of Energy (Contract No. DE-F-G02-88ER13916A000), and benefited from computer time provided by The Ohio Supercomputer Center.

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